logic in color

#3: Generalizing Logic I

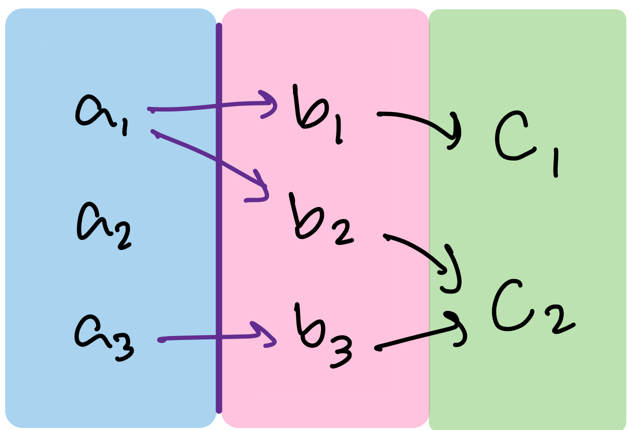
[Matrices & Monads]

Welcome back!

We've been thinking about relations.

A relation is a matrix of truth values.

$$A \xrightarrow{R} B \xrightarrow{S} C \qquad \mathbb{B} = (\{0,1\}, \wedge, \vee)$$



$$\begin{array}{c|ccc} R & b_1 & b_2 & b_3 \\ \hline a_1 & 1 & 1 & 0 \\ a_2 & 0 & 0 & 0 \\ a_3 & 0 & 0 & 1 \end{array} \quad \begin{array}{c|cc} S & c_1 & c_2 \\ \hline b_1 & 1 & 0 \\ b_2 & 0 & 1 \\ b_3 & 0 & 1 \end{array} \quad = \quad \begin{array}{c|cc} & c_1 & c_2 \\ \hline a_1 & 1 & 1 \\ a_2 & 0 & 0 \\ a_3 & 0 & 1 \end{array}$$

Composition is matrix multiplication

$$\Sigma_b R_{ab} \cdot S_{bc} \;:=\; \exists b.\, aRb \wedge bSc$$

& identity is the identity matrix.

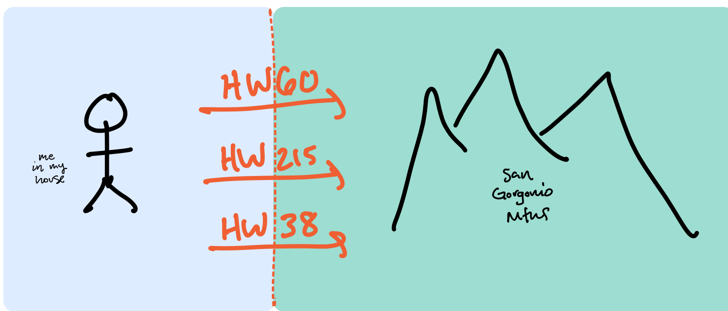
✱ Yet there are many kinds of data which can connect objects. ✱

This is the key to generalize logic.

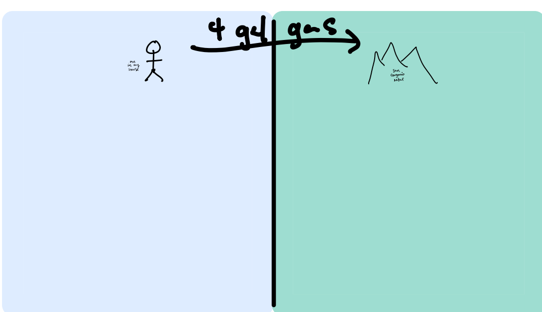
A judgement can contain rich data,
beyond just 0s + 1s:

it could be a distance

or a set of connections, & much more.



What kinds of judgements can we make?



4gal gas (resources, time, work)

modes of transportation

byproducts (sweat, emissions)

Type Theory began with realizing
"whoa, judgements can be more than propositions."
Same here, but we just call it Logic.

The only condition is that the data compose:
as long as we have "matrix multiplication"

$$a \; R \circ S \; c \;=\; \sum b{:}B.\ aRb \cdot bSc \qquad *$$

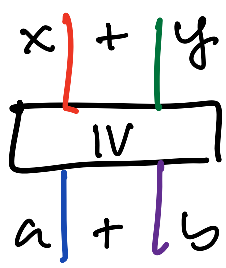
Set: $R \circ S(a,c) = \sum_{b \in B}. R(a,b) \times S(b,c)$

$\mathbb{R}$: $\min_{b \in B}. d_R(a,b) + d_S(b,c)$

then the "kind of judgement" (which we'll call $\mathbb{V}$)
likely forms its own kind of "$\mathbb{V}$-logic"!

→ just one thing: What is the "kind of inference";
what kind of structure is $\mathbb{V}$?

Well, for "a notion of judgement & inference"
$\mathbb{V}$ can be a double category
with just one type & one term.
This is a monoidal category.



multiplication

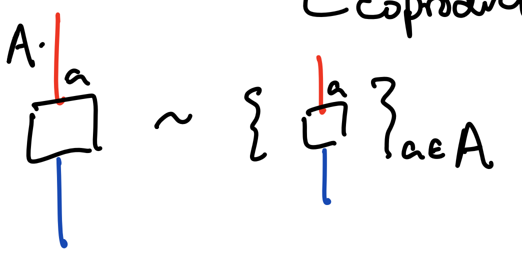
"category version of a rig"

What do we need for matrix multiplication?

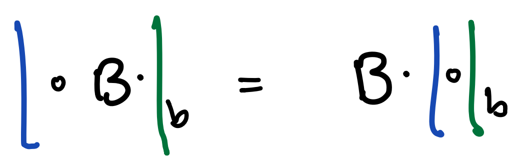
(Set-indexed) sums, which get along with composition.

└ coproducts

(⊗ distributes over sum)



&



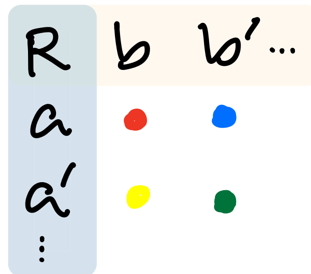
Let's call this condition being **tensored**.

def Let $\mathbb{V}$ be a tensored monoidal category.

Then $\mathbf{Mat}\mathbb{V}$ is a double category:

type $\quad A \quad$ set

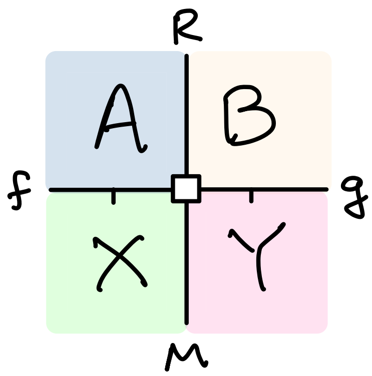
judgement $\quad A \mid B \quad$ matrix $R : A \times B \to \mathbb{V}$

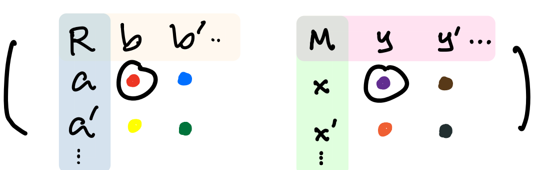


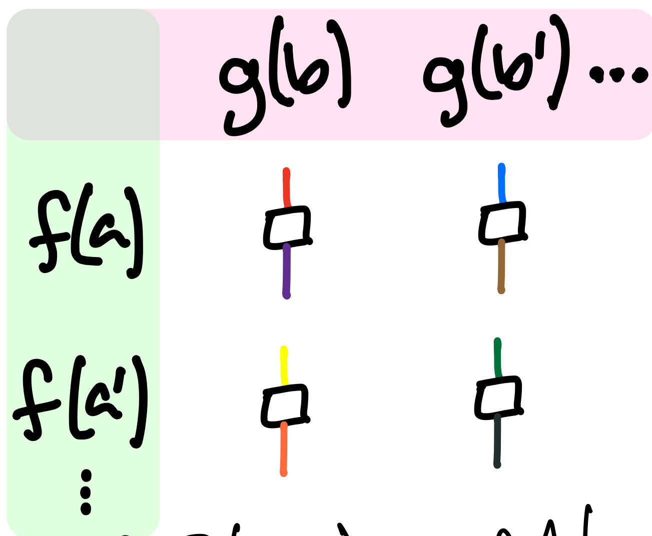
↳ composition $\quad a\, RS\, c = \Sigma b \in B.\; aRb \circ bSc$

↳ identity $\quad a I a' = \begin{cases} 1 & a = a' \\ 0 & a \neq a' \end{cases} \quad \begin{pmatrix} 1 : \text{unique type} \\ 0 : \text{empty sum} \end{pmatrix}$

inference







$\prod_{a \in A,\, b \in B}.\; R(a,b) \to M(a,b)$

↳ sequence + parallel composition. $\square$

$\mathbf{Mat}\mathbb{V}$ is the ground for "$\mathbb{V}$-valued logic."

what is it like for $\mathbb{V} = (\mathrm{Set}, \times, 1)$? $(\mathbb{R}, \leq, +, 0)$? ☆

The world of $\mathrm{Mat}\mathbb{V}$ is nice, $P(X)$

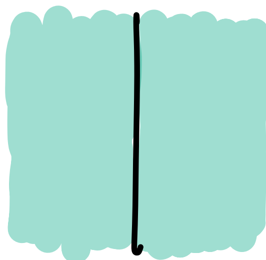
but it's missing essential aspects of logic. (*)

The problem is that the data of judgements is not yet "in" the types — so far just sets. (ex)

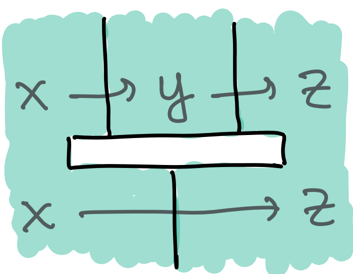
This is an issue even for relations:
plain old sets don't "know about" implication.

* what kind of structures do?

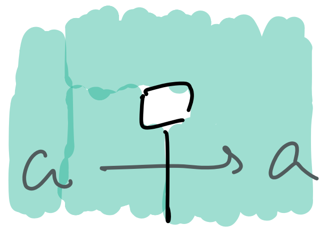
Let R be a relation on a set A.



$A : x \xrightarrow{R} y : A$



composition

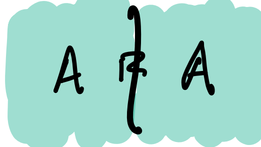


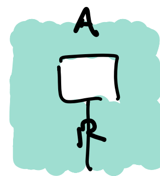
identity

A preorder is a set & relation $A \xrightarrow{R} A$
with implications comp : $R \circ R \Rightarrow R$
& id : $1_A \Rightarrow R$.

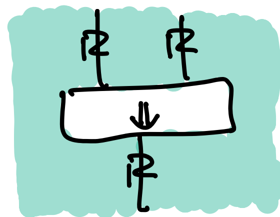
This is a "monad" in Rel.

def Let $\mathbb{C}$ be a double category.

A **monad** in $\mathbb{C}$ is a judgement 

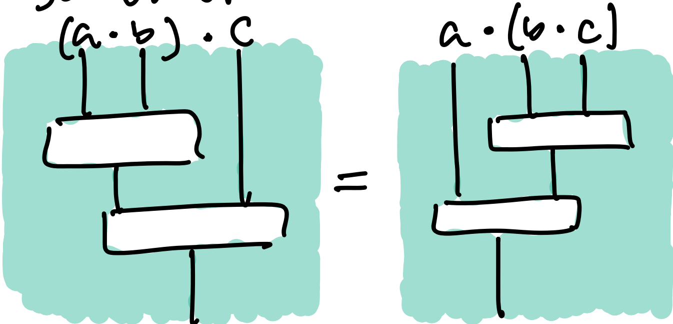
with inferences 

unit

$aRb \cdot bRc \to aRc$ 

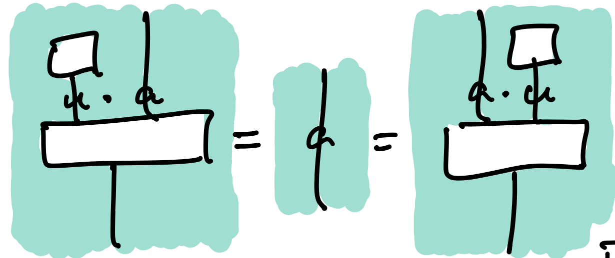
join

so that



assoc

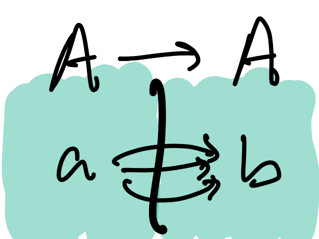
and

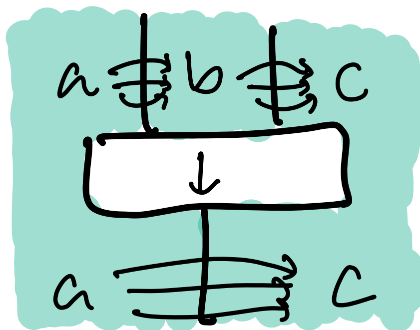


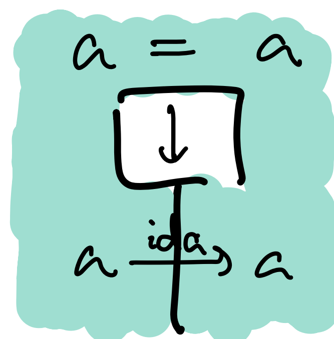
lunit runit ☑

So a monad is "a judgement (string) with composition & identity"

* What's a monad in Mat(Set)? Span

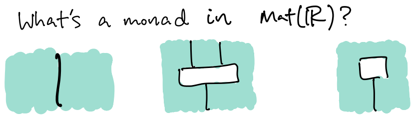






a category!

* What's a monad in Mat($\mathbb{R}$)?



(Lawvere) metric space!

For a double category $\mathbb{C}$ (with one condition)
there's a double category of

"monads + modules" in $\mathbb{C}$,
denoted $\mathrm{Mod}(\mathbb{C})$.

For $\mathbb{V}$-logic, we'll explore $\overline{\mathbb{V}} := \mathrm{Mod}(\mathrm{Mat}(\mathbb{V}))$.
This is a very rich world for generalized logic.

What can we learn + do in this world?

Questions / Thoughts ?

Thanks!